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Monopole condensation and dimensional transmutation in SU(2) QCD

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Abstract: We resolve the controversy regarding the stability of the monopole condensation in the one-loop effective action of SU(2) QCD by calculating the imaginary part of the effective action with two different methods at one-loop order. Our result confirms that the effective action for the magnetic background has no imaginary part but the one for the electric background has a negative imaginary part. This confirms that the monopole condensation is indeed stable, but the electric background becomes unstable due to the pair-annihilation of gluons.

Keywords: Confinement, Gauge Symmetry, QCD.

Contents

1. Introduction

One of the most outstanding problems in theoretical physics is the confinement problem in quantum chromodynamics (QCD). It has long been argued that monopole condensation can explain the confinement of color through the dual Meissner effect [1, 2]. Indeed, if one assumes monopole condensation, one can easily argue that the ensuing dual Meissner effect guarantees the confinement [2, 3]. A satisfactory theoretical proof of the desired monopole condensation in QCD, however, has been very elusive [4, 5, 6]. Fortunately, it has recently been shown that one could demonstrate the monopole condensation by calculating the one-loop effective action in SU(2) QCD [7, 8]. In this paper we present a perturbative method which endorses the stability of the monopole condensation in QCD. Our result strongly indicates that the monopole condensation describes the stable vacuum of QCD, which could guarantee the confinement of color.

A natural way to establish the monopole condensation in QCD is to demonstrate that the quantum fluctuation triggers a dimensional transmutation and phase transition through the Coleman-Weinberg mechanism in QCD. Coleman and Weinberg have demonstrated that quantum effects could trigger a dimensional transmutation in massless scalar QED with a quartic self-interaction of scalar electrons, by showing that the one-loop effective action generates a mass gap through the condensation of scalar electrons which defines a non-trivial new vacuum [9]. To prove the monopole condensation, one must demonstrate such a phase transition in QCD. There have been many attempts to demonstrate monopole condensation in QCD with the one-loop effective action using the background field method [4, 5, 6]. Savvidy was the first to calculate the effective action of SU(2) QCD in the presence of an *ad hoc* color magnetic background, and discovered encouraging evidence for magnetic condensation as a non-trivial vacuum of QCD [4]. But this calculation was repeated by Nielsen and Olesen, who found that the magnetic background generates an imaginary part to the effective action which makes the vacuum unstable [5]. The origin of this instability of the "Savvidy-Nielsen-Olesen (SNO) vacuum" was traced back to the fact that it is not gauge invariant. Because of this the SNO effective action develops an imaginary part which destabilizes the magnetic condensation through the pair creation of gluons. This instability of the SNO vacuum has been widely accepted and never been seriously challenged. This has created the unfortunate impression that it might be impossible to establish the monopole condensation with the effective action of QCD.

A few years ago, however, there was a new attempt to calculate the one-loop effective action of QCD with a gauge independent separation of the classical background from the quantum field [7]. Remarkably, in this calculation the effective action was shown to produce no imaginary part in the presence of the non-abelian monopole background, but a negative imaginary part in the presence of the pure color electric background. This means that in QCD the non-abelian monopole background produces a stable monopole condensation, but the color electric background becomes unstable through the pair annhilation of valence gluons at one-loop level. This remarkable result was obtained by the infra-red regularization of the effective action that respects causality. With this infra-red regularization by causality one has the dual Meissner effect and the magnetic confinement of color in SU(2) QCD [7].

The new result sharply contradicts with earlier results, in particular with respect to the stability of the monopole condensation. This has resurrected the old controversy on the stability of monopole condensation. Since the stability of the monopole condensation is such an important issue for the confinement of color in QCD, it is imperative that one looks for an independent method to resolve the controversy.

A remarkable feature of gauge theories is that in the massless limit the imaginary part of the effective action is proportional to g^2, where g is the gauge coupling constant. This has been shown to be true in both QCD [5]–[8] and massless QED [10, 11]. This allows us to check the presence (or the absence) of the imaginary part by a perturbative method [8]. *The purpose of this paper is to discuss the perturbative method to calculate the imaginary part of the one-loop effective action in massless gauge theories in more detail. We test the validity of this method in massless QED, and apply this to* SU(2) *QCD. We confirm that the QCD effective action indeed has no imaginary part in the presence of the monopole background but has a negative imaginary part in the presence of the color electric background.* This endorses the result based on the infra-red regularization by causality [7], and guarantees the stability of the monopole condensation in QCD.

It should be pointed out that the first to dispute the SNO effective action, and to suggest that the imaginary part of the QCD effective action could be calculated by a perturbative method, was Schanbacher [12]. Unfortunately this remarkable suggestion has met with skepticism and thus been completely ignored. This is partly because that work also lacked the desired gauge invariance but more probably because this suggestion had never been confirmed by a concrete example. In this paper we verify this suggestion in massless QED, and calculate the imaginary part of the QCD effective action perturbatively with two different methods. The first method is to draw the relevant Feynman diagrams and calculate the imaginary part at one-loop order. The second is to apply Schwinger's method of the perturbative expansion [13] of the QED effective action to QCD, and calculate the imaginary part to the second order. These methods reproduce an identical result, identical to what we obtained with the infra-red regularization by causality.

The paper is organized as follows. In section 2 we review the abelian formalism of QCD to make the similarity between QED and QCD more transparent. In section 3 we reproduce the one-loop effective action of QCD using the abelian formalism. In section 4 we compare our effective action with the SNO effective action, and review the origin of the instability of the SNO vacuum. In section 5 we calculate the imaginary part of the effective action of massless QED perturbatively, and demonstrate that the perturbative result produces an identical result, identical to the non-perturbative result. In section 6 we calculate the imaginary part of the QCD effective action perturbatively with the Feynman diagrams. In section 7 we apply Schwinger's method to evaluate the effective action of QCD to reproduce the desired result. In section 8 we discuss the physical implications of our findings. Finally in the appendix we follow Schwinger's method step by step to reconfirm the stability of the monopole condensate.

2. Abelian formalism of QCD

In this section we review the gauge-independent abelianization of QCD [2, 3], which we need to calculate the imaginary part of the effective action using Schwinger's method. We will concentrate on SU(2) QCD in this paper, for simplicity. We start from the gauge-independent decomposition of the gauge potential into the restricted potential $\hat{A}_\mu$ and the valence potential $\vec{X}_\mu$ [2, 3], which has recently been dubbed the "Cho decomposition" or "Cho-Faddeev-Niemi decomposition" [14, 15, 16]. Let $\hat{n}$ be the unit isovector which selects the color charge direction everywhere in space-time, and let [2, 3]

$$\vec{A}_\mu = A_\mu \hat{n} - \frac{1}{g}\hat{n} \times \partial_\mu \hat{n} + \vec{X}_\mu = \hat{A}_\mu + \vec{X}_\mu \,,$$
$$(A_\mu = \hat{n}\cdot\vec{A}_\mu \,, \qquad \hat{n}^2 = 1 \,, \qquad \hat{n}\cdot\vec{X}_\mu = 0) \,, \tag{2.1}$$

where A_μ is the "electric" potential. Notice that the restricted potential $\hat{A}_\mu$ is precisely the connection which leaves $\hat{n}$ invariant under the parallel transport,

$$\hat{D}_\mu \hat{n} = \partial_\mu \hat{n} + g\hat{A}_\mu \times \hat{n} = 0 \,. \tag{2.2}$$

Under the infinitesimal gauge transformation

$$\delta\hat{n} = -\vec{\alpha}\times\hat{n} \,, \qquad \delta\vec{A}_\mu = \frac{1}{g}D_\mu\vec{\alpha} \,, \tag{2.3}$$

one has

$$\begin{aligned} \delta A_\mu &= \frac{1}{g}\hat{n}\cdot\partial_\mu\vec{\alpha} \,, \qquad \delta\hat{A}_\mu = \frac{1}{g}\hat{D}_\mu\vec{\alpha} \,, \\ \delta\vec{X}_\mu &= -\vec{\alpha}\times\vec{X}_\mu \,. \end{aligned} \tag{2.4}$$

This tells us that $\hat{A}_\mu$ by itself describes an SU(2) connection which enjoys the full SU(2) gauge degrees of freedom. Furthermore the valence potential $\vec{X}_\mu$ forms a gauge covariant vector field under the gauge transformation. *But what is really remarkable is that the decomposition is gauge independent. Once the gauge covariant topological field $\hat{n}$ is chosen, the decomposition follows automatically, regardless of the choice of gauge* [2, 3].

Remember that $\hat{A}_\mu$ retains all the essential topological characteristics of the original non-abelian potential. Clearly $\hat{n}$ defines $\pi_2(S^2)$ which describes the non-abelian monopoles [17, 18, 19], and characterizes the Hopf invariant $\pi_3(S^2) \simeq \pi_3(S^3)$ which describes the topologically distinct vacua [20, 21]. Furthermore $\hat{A}_\mu$ has a dual structure,

$$\begin{aligned}
\hat{F}_{\mu\nu} &= \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + g \hat{A}_\mu \times \hat{A}_\nu = (F_{\mu\nu} + H_{\mu\nu})\hat{n}\,, \\
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu\,, \\
H_{\mu\nu} &= -\frac{1}{g}\hat{n} \cdot (\partial_\mu \hat{n} \times \partial_\nu \hat{n}) = \partial_\mu \tilde{C}_\nu - \partial_\nu \tilde{C}_\mu\,,
\end{aligned} \tag{2.5}$$

where $\tilde{C}_\mu$ is the "magnetic" potential of the monopoles (Notice that one can always introduce the magnetic potential since $H_{\mu\nu}$ forms a closed two-form locally sectionwise) [2, 3]. Thus, one can identify the non-abelian monopole potential by

$$\vec{C}_\mu = -\frac{1}{g}\hat{n} \times \partial_\mu \hat{n}\,, \tag{2.6}$$

in terms of which the magnetic field is expressed by

$$\vec{H}_{\mu\nu} = \partial_\mu \vec{C}_\nu - \partial_\nu \vec{C}_\mu + g \vec{C}_\mu \times \vec{C}_\nu = H_{\mu\nu}\hat{n}\,. \tag{2.7}$$

This provides the gauge independent separation of the monopole field $H_{\mu\nu}$ from the color electromagnetic field $F_{\mu\nu}$. The monopole potential (2.6) has been referred to as the "Cho connection" [14, 15].

With the decomposition (2.1), one has

$$\vec{F}_{\mu\nu} = \hat{F}_{\mu\nu} + \hat{D}_\mu \vec{X}_\nu - \hat{D}_\nu \vec{X}_\mu + g \vec{X}_\mu \times \vec{X}_\nu\,, \tag{2.8}$$

so that the Yang-Mills lagrangian is expressed as

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}\vec{F}_{\mu\nu}^2 \\
&= -\frac{1}{4}\hat{F}_{\mu\nu}^2 - \frac{1}{4}(\hat{D}_\mu \vec{X}_\nu - \hat{D}_\nu \vec{X}_\mu)^2 - \frac{g}{2}\hat{F}_{\mu\nu} \cdot (\vec{X}_\mu \times \vec{X}_\nu) - \frac{g^2}{4}(\vec{X}_\mu \times \vec{X}_\nu)^2\,.
\end{aligned} \tag{2.9}$$

This shows that the Yang-Mills theory can be viewed as a restricted gauge theory made of the restricted potential, which has the valence gluons as its source [2, 3].

An important advantage of the decomposition (2.1) is that it can actually abelianize (or more precisely "dualize") the non-abelian gauge theory [2, 3]. To see this let $(\hat{n}_1,\ \hat{n}_2,\ \hat{n})$ be a right-handed orthonormal basis and let

$$\vec{X}_\mu = X^1_\mu\ \hat{n}_1 + X^2_\mu\ \hat{n}_2\,, \qquad (X^1_\mu = \hat{n}_1 \cdot \vec{X}_\mu\,, \qquad X^2_\mu = \hat{n}_2 \cdot \vec{X}_\mu)$$

and find

$$\hat{D}_\mu \vec{X}_\nu = \Big[\partial_\mu X^1_\nu - g(A_\mu + \tilde{C}_\mu) X^2_\nu\Big]\hat{n}_1 + \Big[\partial_\mu X^2_\nu + g(A_\mu + \tilde{C}_\mu) X^1_\nu\Big]\hat{n}_2 \,. \tag{2.10}$$

So with

$$B_\mu = A_\mu + \tilde{C}_\mu \,, \qquad X_\mu = \frac{1}{\sqrt{2}}(X^1_\mu + iX^2_\mu) \,, \tag{2.11}$$

one could express the lagrangian explicitly in terms of the dual potential B_μ and the complex vector field X_μ,

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}G^2_{\mu\nu} - \frac{1}{2}|\hat{D}_\mu X_\nu - \hat{D}_\nu X_\mu|^2 + igG_{\mu\nu}X^*_\mu X_\nu - \frac{1}{2}g^2[(X^*_\mu X_\mu)^2 - (X^*_\mu)^2(X_\nu)^2] \\ &= -\frac{1}{4}(G_{\mu\nu} + X_{\mu\nu})^2 - \frac{1}{2}|\hat{D}_\mu X_\nu - \hat{D}_\nu X_\mu|^2 \,, \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} G_{\mu\nu} &= F_{\mu\nu} + H_{\mu\nu} \,, \qquad \hat{D}_\mu X_\nu = (\partial_\mu + igB_\mu)X_\nu \,, \\ X_{\mu\nu} &= -ig(X^*_\mu X_\nu - X^*_\nu X_\mu) \,. \end{aligned}$$

Clearly this describes an abelian gauge theory coupled to the charged vector field X_μ. But the important point here is that the abelian potential B_μ is given by the sum of the electric and magnetic potentials $A_\mu + \tilde{C}_\mu$. In this form the equations of motion of SU(2) QCD are expressed by

$$\begin{aligned} \partial_\mu(G_{\mu\nu} + X_{\mu\nu}) &= igX^*_\mu(\hat{D}_\mu X_\nu - \hat{D}_\nu X_\mu) - igX_\mu(\hat{D}_\mu X_\nu - \hat{D}_\nu X_\mu)^* \,, \\ \hat{D}_\mu(\hat{D}_\mu X_\nu - \hat{D}_\nu X_\mu) &= igX_\mu(G_{\mu\nu} + X_{\mu\nu}) \,. \end{aligned} \tag{2.13}$$

This shows that one can indeed abelianize the non-abelian theory with our decomposition. The remarkable change in this abelian formulation is that here the topological field $\hat{n}$ is replaced by the magnetic potential $\tilde{C}_\mu$ [2, 3].

An important feature of this abelianization is that it is gauge independent, because here we have never fixed the gauge to obtain this abelian formalism. So one might ask how the non-abelian gauge symmetry is realized in this abelian formalism. To discuss this let

$$\begin{aligned} \vec{\alpha} &= \alpha_1\ \hat{n}_1 + \alpha_2\ \hat{n}_2 + \theta\ \hat{n} \,, \qquad \alpha = \frac{1}{\sqrt{2}}(\alpha_1 + i\ \alpha_2) \,, \\ \vec{C}_\mu &= -\frac{1}{g}\hat{n} \times \partial_\mu \hat{n} = -C^1_\mu \hat{n}_1 - C^2_\mu \hat{n}_2 \,, \qquad C_\mu = \frac{1}{\sqrt{2}}(C^1_\mu + i\ C^2_\mu) \,. \end{aligned} \tag{2.14}$$

Then the lagrangian (2.12) is invariant not only under the active gauge transformation (2.4) described by

$$\begin{aligned} \delta A_\mu &= \frac{1}{g}\partial_\mu\theta - i(C^*_\mu\alpha - C_\mu\alpha^*) \,, \qquad \delta\tilde{C}_\mu = -\delta A_\mu \,, \\ \delta X_\mu &= 0 \,, \end{aligned} \tag{2.15}$$

but also under the following passive gauge transformation described by

$$\begin{aligned}\delta A_\mu &= \frac{1}{g}\partial_\mu\theta - i(X^*_\mu\alpha - X_\mu\alpha^*)\,, \qquad \delta\tilde{C}_\mu = 0\,,\\ \delta X_\mu &= \frac{1}{g}\hat{D}_\mu\alpha - i\theta X_\mu\,.\end{aligned} \tag{2.16}$$

Clearly this passive gauge transformation ensures the desired non-abelian gauge symmetry for the abelian formalism. This means that the abelian theory not only retains the original gauge symmetry, but actually has an additional (both the active and passive) gauge symmetry. *But we emphasize that this is not the "naive" abelianization of QCD which one obtains by fixing the gauge. Our abelianization is a gauge-independent abelianization. Besides, here the abelian gauge group is* $\mathrm{U}(1)_e\otimes\mathrm{U}(1)_m$*, so that the theory becomes a dual gauge theory* [2, 3]. This is evident from (2.15) and (2.16). This abelianized QCD will become important in the following.

3. Monopole condensation: a review

To demonstrate the validity of the abelian formalism we now calculate the effective action of QCD with the abelian formalism, and reproduce the same effective action that we obtained with the non-abelian formalism. So, using the background field method [22, 23], we first divide the gluon field into two parts, the slow-varying classical part B_μ and the fluctuating quantum part X_μ, and identify B_μ as the classical background. In this picture the active gauge transformation (2.15) is viewed as the background gauge transformation and the passive gauge transformation (2.16) is viewed as the quantum gauge transformation. Now, we fix the gauge of the quantum gauge transformation by imposing the following gauge condition on X_μ,

$$\hat{D}_\mu X_\mu = 0\,, \qquad (\hat{D}_\mu X_\mu)^* = 0\,, \qquad \mathcal{L}_{gf} = -\frac{1}{\xi}|\hat{D}_\mu X_\mu|^2\,. \tag{3.1}$$

Remarkably, under the gauge transformation (2.16) the gauge condition depends only on α, so the corresponding Faddeev-Popov determinant is given by

$$M_{FP} = \begin{vmatrix} \frac{\delta(\hat{D}_\mu X_\mu)}{\delta\alpha} & \frac{\delta(\hat{D}_\mu X_\mu)}{\delta\alpha^*} \\ \frac{\delta(\hat{D}_\mu X_\mu)^*}{\delta\alpha} & \frac{\delta(\hat{D}_\mu X_\mu)^*}{\delta\alpha^*} \end{vmatrix}\,. \tag{3.2}$$

With this gauge fixing the effective action takes the following form,

$$\begin{aligned}\exp\Big[iS_{\text{eff}}(B_\mu)\Big] &= \int \mathcal{D}X_\mu\mathcal{D}X^*_\mu\mathcal{D}c_1\mathcal{D}c_1^\dagger\mathcal{D}c_2\mathcal{D}c_2^\dagger\,\times\\ &\times\exp\Big\{i\int\Big[-\frac{1}{4}(G_{\mu\nu}+X_{\mu\nu})^2 - \frac{1}{2}|\hat{D}_\mu X_\nu - \hat{D}_\nu X_\mu|^2 - \frac{1}{\xi}|\hat{D}_\mu X_\mu|^2 +\\ &\qquad + c_1^\dagger(\hat{D}^2 + g^2X^*_\mu X_\mu)c_1 - g^2c_1^\dagger X_\mu X_\mu c_2 +\\ &\qquad + c_2^\dagger(\hat{D}^2+g^2X^*_\mu X_\mu)^*c_2 - g^2c_2^\dagger X^*_\mu X^*_\mu c_1\Big]d^4x\Big\}\,,\end{aligned} \tag{3.3}$$

where c_1 and c_2 are the complex ghost fields.

To evaluate the integral notice that the functional determinants of the valence gluon and the ghost loops are expressed as

$$\mathrm{Det}^{-\frac{1}{2}} K_{\mu\nu} \simeq \mathrm{Det}[-g_{\mu\nu}(\hat{D}\hat{D}) + 2igG_{\mu\nu}]\,, \qquad \mathrm{Det} M_{FP} = \mathrm{Det}[-(\hat{D}\hat{D})]^2\,. \tag{3.4}$$

Using the relation

$$G_{\mu\alpha}G_{\nu\beta}G_{\alpha\beta} = \frac{1}{2}G^2 G_{\mu\nu} + \frac{1}{2}(G\tilde{G})\tilde{G}_{\mu\nu}\,, \qquad (\tilde{G}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}G_{\rho\sigma})\,, \tag{3.5}$$

one can simplify the functional determinants of the gluon and the ghost loops as follows,

$$\begin{aligned} \ln \mathrm{Det}^{-\frac{1}{2}} K &= \ln \mathrm{Det}[(-\hat{D}^2 + 2a)(-\hat{D}^2 - 2a)] + \ln \mathrm{Det}[(-\hat{D}^2 - 2ib)(-\hat{D}^2 + 2ib)]\,, \\ \ln \mathrm{Det} M_{FP} &= 2\ln \mathrm{Det}(-\hat{D}^2)\,, \end{aligned} \tag{3.6}$$

where

$$a = \frac{g}{2}\sqrt{\sqrt{G^4 + (G\tilde{G})^2} + G^2}\,, \qquad b = \frac{g}{2}\sqrt{\sqrt{G^4 + (G\tilde{G})^2} - G^2}\,.$$

With this one has [7, 8]

$$\Delta S = i\ln \mathrm{Det}[(-\hat{D}^2 + 2a)(-\hat{D}^2 - 2a)] + i\ln \mathrm{Det}[(-\hat{D}^2 - 2ib)(-\hat{D}^2 + 2ib)] - 2i\ln \mathrm{Det}(-\hat{D}^2)\,. \tag{3.7}$$

One can evaluate the functional determinants and find

$$\begin{aligned} \Delta\mathcal{L} = &\frac{1}{16\pi^2}\int_0^\infty \frac{dt}{t^{3-\epsilon}} \frac{abt^2/\mu^4}{\sinh(at/\mu^2)\sin(bt/\mu^2)} \times \\ &\times\left[\exp\left(-\frac{2at}{\mu^2}\right) + \exp\left(\frac{2at}{\mu^2}\right) + \exp\left(\frac{2ibt}{\mu^2}\right) + \exp\left(-\frac{2ibt}{\mu^2}\right) - 2\right], \end{aligned} \tag{3.8}$$

where μ is a dimensional parameter. This is exactly the same expression of the effective action obtained before, with the non-abelian formalism of QCD [4, 6, 7, 8]. This demonstrates that our abelian formalism is identical to the conventional non-abelian formalism of QCD.

The evaluation of the integral (3.8) for arbitrary a and b has been notoriously difficult [4, 5, 6]. Even in the case of "simpler" QED the integration of the effective action has been completed only recently [10, 11]. Fortunately the integral for the pure electric ($a = 0$) and pure magnetic ($b = 0$) background has now been correctly performed [7, 8]. For a pure monopole background (i.e., with $G_{\mu\nu} = H_{\mu\nu}$) we have $b = 0$, so that the integral (3.8) becomes

$$\Delta\mathcal{L} = \frac{1}{16\pi^2}\int_0^\infty \frac{dt}{t^2}\frac{a/\mu^2}{\sinh(at/\mu^2)}\left[\exp\left(-\frac{2at}{\mu^2}\right) + \exp\left(+\frac{2at}{\mu^2}\right)\right]. \tag{3.9}$$

To perform the integral we have to regularize the infra-red divergence first. For this we go to the Minkowski time with the Wick rotation, and find [7, 8]

$$\begin{aligned} \Delta\mathcal{L} &= \Delta\mathcal{L}_+ + \Delta\mathcal{L}_-\,, \\ \Delta\mathcal{L}_+ &= -\frac{1}{16\pi^2}\int_0^\infty \frac{dt}{t^2}\frac{a/\mu^2}{\sin(at/\mu^2)}\exp\left(-\frac{2iat}{\mu^2}\right), \\ \Delta\mathcal{L}_- &= -\frac{1}{16\pi^2}\int_0^\infty \frac{dt}{t^2}\frac{a/\mu^2}{\sin(at/\mu^2)}\exp\left(+\frac{2iat}{\mu^2}\right). \end{aligned} \tag{3.10}$$

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In this form the infra-red divergence has disappeared, but now we face an ambiguity in choosing the correct contours of the integrals in (3.10). Fortunately this ambiguity can be resolved by causality. To see this notice that the two integrals $\Delta\mathcal{L}_+$ and $\Delta\mathcal{L}_-$ originate from the two determinants $\mathrm{Det}(-\hat{D}^2+2a)$ and $\mathrm{Det}(-\hat{D}^2-2a)$, and the standard causality argument requires us to identify $2a$ in the first determinant as $2a-i\epsilon$ but in the second determinant as $2a+i\epsilon$. *This tells that the poles in the first integral in (3.10) should lie above the real axis, but the poles in the second integral should lie below the real axis. From this we conclude that the contour in $\Delta\mathcal{L}_+$ should pass below the real axis, but the contour in $\Delta\mathcal{L}_-$ should pass above the real axis.* With this causality requirement the two integrals become complex conjugate to each other. This guarantees that $\Delta\mathcal{L}$ is explicitly real, with no imaginary part.

With this infra-red regularization by causality we obtain

$$\begin{aligned}\mathcal{L}_{\text{eff}} &= -\frac{a^2}{2g^2} - \frac{11}{48\pi^2}a^2\left(\ln\frac{a}{\mu^2} - c\right),\\ c &= 1 - \ln 2 - \frac{24}{11}\zeta'\left(-1,\frac{3}{2}\right) = 0.94556\ldots\,,\end{aligned} \tag{3.11}$$

for a pure monopole background, where $\zeta(x,y)$ is the generalized Hurwitz zeta function [7, 8].

For the pure electric background (i.e., with $a=0$) the infra-red regularization by causality gives us [7, 8]

$$\mathcal{L}_{\text{eff}} = \frac{b^2}{2g^2} + \frac{11b^2}{48\pi^2}\left(\ln\frac{b}{\mu^2} - c\right) - i\frac{11b^2}{96\pi}\,. \tag{3.12}$$

Observe that (3.11) and (3.12) are related by the duality. In fact we can obtain one from the other simply by replacing a with $-ib$ and b with ia. This duality, which states that the effective action should be invariant under the substitution

$$a \to -ib\,, \qquad b \to ia\,, \tag{3.13}$$

was recently established as a fundamental symmetry of the effective action of gauge theory, both abelian and non-abelian [7, 11]. We emphasize that the duality provides a very useful tool to check the self-consistency of the effective action.

Remarkably the effective action (3.11) generates the much desired dimensional transmutation in QCD, the phenomenon Coleman and Weinberg first observed in massless scalar QED [9]. It is this dimensional transmutation that produces the dynamical generation of a mass gap through the monopole condensation [7, 8]. To demonstrate this we first renormalize the effective action. Notice that the effective action (3.11) provides the following effective potential

$$V = \frac{1}{2}\frac{a^2}{g^2}\left[1 + \frac{11g^2}{24\pi^2}\left(\ln\frac{a}{\mu^2} - c\right)\right]. \tag{3.14}$$

So we define the running coupling $\bar{g}$ by [4, 7]

$$\left.\frac{\partial^2 V}{\partial a^2}\right|_{a=\bar{\mu}^2} = \frac{1}{\bar{g}^2}\,. \tag{3.15}$$

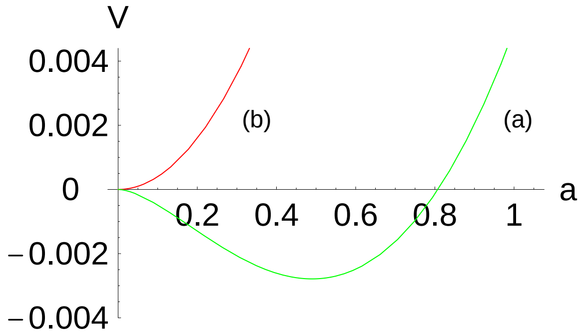

Figure 1: The effective potential of SU(2) QCD in the pure magnetic background. Here (a) is the effective potential and (b) is the classical potential.

With this definition we find

$$\frac{1}{\bar{g}^2} = \frac{1}{g^2} + \frac{11}{24\pi^2}\left(\ln\frac{\bar{\mu}^2}{\mu^2} - c + \frac{3}{2}\right), \tag{3.16}$$

from which we obtain the following β-function,

$$\beta(\bar{\mu}) = \bar{\mu}\frac{\partial \bar{g}}{\partial \bar{\mu}} = -\frac{11}{24\pi^2}\bar{g}^3 \,. \tag{3.17}$$

This is exactly the same β-function obtained from perturbative QCD to prove asymptotic freedom [24]. This confirms that our effective action is consistent with the asymptotic freedom.

The fact that the β-function obtained from the effective action becomes identical to the one obtained by the perturbative calculation is perhaps what one might have expected. But we emphasize that this is not always the case. In fact in QED it has been demonstrated that the running coupling and the β-function obtained from the effective action is different from those obtained from the perturbative method [7, 11].

In terms of the running coupling the renormalized potential is given by

$$V_{\rm ren} = \frac{1}{2}\frac{a^2}{\bar{g}^2}\left[1 + \frac{11}{24\pi^2}\bar{g}^2\left(\ln\frac{a}{\bar{\mu}^2} - \frac{3}{2}\right)\right], \tag{3.18}$$

which generates a non-trivial local minimum at

$$\langle a \rangle = \bar{\mu}^2 \exp\left(-\frac{24\pi^2}{11\bar{g}^2} + 1\right). \tag{3.19}$$

Notice that with $\bar{\alpha}_s = 1$ we have

$$\frac{\langle a \rangle}{\bar{\mu}^2} = 0.48988\ldots \,. \tag{3.20}$$

This is nothing but the desired magnetic condensation. This proves that the one loop effective action of QCD in the presence of the constant magnetic background does generate a phase transition through the monopole condensation [7, 8].

The corresponding effective potential is plotted in figure 1, where we have assumed $\bar{\alpha}_s = 1$, $\bar{\mu} = 1$. The effective potential clearly shows that there is indeed a dynamical generation of mass gap which indicates the existence of the confinement phase in QCD.

4. Stability of monopole condensation

The effective action of QCD in the presence of a pure magnetic background (i.e., for $b = 0$) was first calculated by Savvidy and subsequently by Nielsen and Olesen [4, 5]. Their effective action was almost identical to ours, except that theirs contains the following extra imaginary part,

$$\mathrm{Im}\ \mathcal{L}_{\mathrm{eff}}\Big|_{SNO} = \frac{a^2}{8\pi}\,, \qquad b = 0\,. \tag{4.1}$$

This sharply contradicts with our result (3.11) and (3.12), which has the following imaginary part,

$$\mathrm{Im}\,\mathcal{L}_{\mathrm{eff}} = \begin{cases} 0 & b = 0 \\ -\dfrac{11b^2}{96\pi} & a = 0\,. \end{cases} \tag{4.2}$$

Clearly the imaginary part of the SNO effective action destabilizes our vacuum (3.19) through the pair creation of gluons. This would nullify our result, the monopole condensation and the dynamical generation of mass in QCD. So it becomes a most urgent issue to find out which of the effective actions is correct.

To settle this issue, one must understand the origin of the difference. The argument for the imaginary part in the SNO effective action was that the magnetic background should generate unstable tachyonic modes in the long distance region. These tachyonic modes, they argued, would generate an imaginary part to the effective action and render the vacuum condensation unstable. And indeed one does obtain (4.1), if one naively regularizes the infra-red divergence of the integral (3.9) with the ζ-function regularization [5, 6].

Notice that these unstable modes are unphysical ones which violate causality. Furthermore, the SNO vacuum is not gauge invariant, and one can argue that the unstable modes originate from this fact. So one must be careful to exclude the unphysical modes when one regularizes the infra-red divergence. As we have pointed out, the infra-red regularization by causality naturally removes the unphysical modes and gives us the correct effective action. Furthermore, this regularization has been shown to preserve the duality, an important consistency condition for the effective actions in gauge theories [7, 8].

The infra-red regularization by causality assures the existence of a stable monopole condensation in SU(2) QCD. On the other hand, since the ζ-function regularization is also a well-established regularization method, it is not at all clear which regularization is the correct one for QCD at this moment. Given the fact that the stability of the vacuum condensation is such an important issue which can make or break the confinement, one must find an independent method to resolve the issue. We now discuss a straightforward perturbative method to calculate the imaginary part of the effective action which can resolve the controversy and guarantee the stability of the monopole condensation. We present two independent methods based on the perturbative expansion, the Feynman diagram calculation and the perturbative calculation of the imaginary part of the effective action, to justify the infra-red regularization by causality.

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5. Imaginary part of effective action in massless QED

Before we discuss QCD we have to show that one can indeed calculate the imaginary part of the effective action either perturbatively or non-perturbatively in massless gauge theories. As far as we know, this has never been demonstrated before. So in this section we first calculate the imaginary part of the effective action of massless QED perturbatively, and show that the perturbative result is identical to the non-perturbative result.

The imaginary part of the non-perturbative one-loop effective action of QED is known to be [10, 11],

$$\mathrm{Im}\,\mathcal{L}_{QED} = \frac{ab}{8\pi^2}\sum_{n=1}^{\infty}\frac{1}{n}\coth\left(\frac{n\pi a}{b}\right)\exp\left(-\frac{n\pi m^2}{b}\right), \tag{5.1}$$

where m is the electron mass and

$$a = \frac{e}{2}\sqrt{\sqrt{F^4+(F\tilde{F})^2}+F^2}\,, \qquad b = \frac{e}{2}\sqrt{\sqrt{F^4+(F\tilde{F})^2}-F^2}\,.$$

In the massless limit this gives us [10, 11]

$$\mathrm{Im}\,\mathcal{L}_{QED}\Big|_{m=0} = \begin{cases} 0\,, & b=0 \\ \dfrac{b^2}{48\pi}\,, & a=0\,. \end{cases} \tag{5.2}$$

Notice that the imaginary part is indeed of the order e^2, which is not the case when $m \neq 0$. We emphasize that the massless limit is well-defined because the one-loop effective action of massless QED has no infra-red divergence when $ab = 0$ [10, 11].

Now we calculate the imaginary part perturbatively to the order e^2. There are two different ways to do this, by calculating the Feynman diagram directly to the order e^2 or by making the perturbative expansion of the non-perturbative one-loop effective action. We start with the Feynman diagram. It is well-known that, to the order e^2, the electron loop diagram for an arbitrary background A_μ (with the dimensional regularization) gives us [23]

$$\begin{aligned}
\frac{1}{2}\int d^4p A_\mu(p)\Pi_{\mu\nu}(p)A_\nu(-p) &= \frac{ie^2}{4\pi^2}\int d^4p A_\mu(p)(p^2 g_{\mu\nu}-p_\mu p_\nu)A_\nu(-p)\,\times \\
&\quad\times\int_0^1 dx x(1-x)\left(-\frac{2}{\epsilon}+\gamma+\ln\left[1-x(1-x)\frac{p^2}{m^2}\right]\right) \\
&= \frac{ie^2}{8\pi^2}\int d^4p F_{\mu\nu}(p)F_{\mu\nu}(-p)\int_0^1 dx x(1-x)\,\times \\
&\quad\times\left(-\frac{2}{\epsilon}+\gamma+\ln\left[1-x(1-x)\frac{p^2}{m^2}\right]\right),
\end{aligned} \tag{5.3}$$

where $\Pi_{\mu\nu}(p)$ is the vacuum polarization tensor and m is the electron mass. In the massless limit (with the subtraction of infra-red divergence at μ^2) this gives us the leading contribution to the effective action

$$\Delta S_{QED} = \frac{e^2}{48\pi^2}\int d^4p F_{\mu\nu}(p)F_{\mu\nu}(-p)\left[\ln\left(\frac{p^2}{\mu^2}\right)+c_0\right], \tag{5.4}$$

where C_0 is a regularization-dependent constant which can be absorbed by the wave function renormalization.

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Clearly the imaginary part could only arise from the term $\ln(p^2/\mu^2)$, so that for a space-like p^2 (with $\mu^2 > 0$) the effective action has no imaginary part. But since a space-like p^2 corresponds to a magnetic background ($b = 0$), we find that the magnetic condensation generates no imaginary part, at least at the order e^2.

As for the electric background ($a = 0$) in general we need to consider a time-like p^2, but here we are interested in a constant electric background. In this case we have to go to the limit $p^2 = 0$ and make the analytic continuation of (5.4) to $p^2 = 0$. This is a delicate task. Fortunately the causality, with the familiar prescription $p^2 \to p^2 - i\epsilon$, guides us to make the analytic continuation correctly. Indeed the causality tells that

$$\lim_{p^2 \to 0} \ln\Big(\frac{p^2}{\mu^2}\Big) \to \lim_{p^2 \to 0} \ln\Big(\frac{p^2}{\mu^2} - i\epsilon\Big) = \infty - i\frac{\pi}{2}\,. \tag{5.5}$$

From this we obtain (to the order e^2)

$$\mathrm{Im}\,\mathcal{L}_{QED}\Big|_{m=0} = \begin{cases} 0\,, & b = 0\,, \\ \dfrac{b^2}{48\pi}\,, & a = 0\,, \end{cases}$$

which obviously is identical to the non-perturbative result (5.2).

Next, we calculate the imaginary part of the effective action of massless QED by making the perturbative expansion of the effective action to the order e^2. For this it is instructive for us to review Schwinger's perturbative calculation of the QED effective action, who obtained [13]

$$\Delta S_{QED} = \frac{e^2}{16\pi^2} \int d^4p F_{\mu\nu}(p) F_{\mu\nu}(-p) \int_0^1 dv \frac{v^2(1 - v^2/3)}{1 - v^2 + 4m^2/p^2}\,. \tag{5.6}$$

Clearly the imaginary part can only arise when the integrand develops a pole, so that the effective action has no imaginary part when $p^2 > -4m^2$. This confirms that, for the magnetic background, the QED effective action has no imaginary part.

But notice that when $p^2 < -4m^2$ the integrand develops a pole at $v^2 = 1 + 4m^2/p^2$, and Schwinger taught us how to calculate the imaginary part of the effective action with the causality prescription $p^2 \to p^2 - i\epsilon$. But we notice that in the massless limit, the pole moves to $v = 1$. In this case the pole contribution to the imaginary part is reduced by a half because only one quarter, not a half, of the pole contributes to the integral. With this observation we again reproduce (5.2) [8].

This proves that one can calculate the imaginary part of the effective action of massless QED perturbatively or non-perturbatively, with identical results. As far as we understand the perturbative calculation of the imaginary part of the effective action of massless QED has not been performed before.

6. Diagrammatic calculation of imaginary part of QCD effective action

A remarkable point of QCD is that the imaginary part of the one-loop effective action is of the order g^2. This is evident from both (4.1) and (4.2). This allows us to calculate the imaginary part perturbatively. In this section we calculate the imaginary part of the

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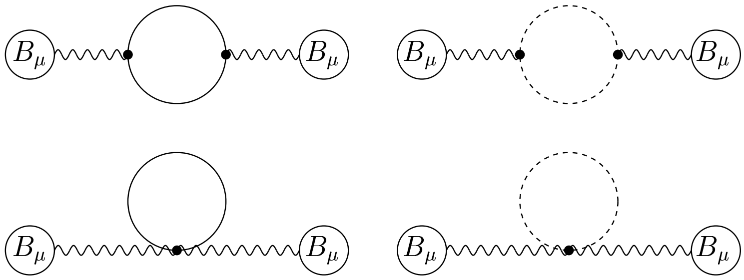

Figure 2: The Feynman diagrams that contribute to the effective action at g^2 order.

effective action of QCD perturbatively from Feynman diagrams. To do this we can either use the non-abelian formalism or the abelian formalism of QCD. Here we will use the latter in order to make the connection between QED and QCD more transparent.

For an arbitrary background B_μ there are four Feynman diagrams that contribute to the order g^2. They are shown schematically in figure 2. Here the straight line and the dotted line represent the valence gluon and the ghost, respectively. The tadpole diagrams contain a quadratic divergence which does not appear in the final result.

The sum of these diagrams is given (in the Feynman gauge with dimensional regularization) by [23]

$$\begin{aligned}\frac{1}{2}\int d^4pB_\mu(p)\Pi_{\mu\nu}(p)B_\nu(-p) &= i\frac{11g^2}{96\pi^2}\int d^4pG_{\mu\nu}(p)G_{\mu\nu}(-p)\int_0^1 dxx(1-x)\times \\ &\times\Big(-\frac{2}{\epsilon}+\gamma+\ln[1-x(1-x)\frac{p^2}{\mu^2}]\Big)\,, \end{aligned} \tag{6.1}$$

so that we have

$$\Delta S_{\text{eff}} = -\frac{11g^2}{96\pi^2}\int d^4pG_{\mu\nu}(p)G_{\mu\nu}(-p)\left[\frac{2}{\epsilon}-\gamma-\ln\left(\frac{p^2}{\mu^2}\right)\right]. \tag{6.2}$$

Evidently the imaginary part can only arise from the term $\ln(p^2/\mu^2)$, and just as in massless QED we have no imaginary part for the monopole condensation (for $b=0$).

To evaluate the imaginary part for a constant electric background (for $a=0$) we have to make the analytic continuation of (6.2) to $p^2=0$. Here again the causality requirement (5.5) dictates how to make the analytic continuation, and we obtain

$$\operatorname{Im}\Delta\mathcal{L} = -\frac{11b^2}{96\pi}\,. \tag{6.3}$$

So (6.2) gives us [8]

$$\operatorname{Im}\mathcal{L}_{\text{eff}} = \begin{cases} 0\,, & b=0 \\ -\dfrac{11b^2}{96\pi}\,, & a=0\,, \end{cases}$$

which is identical to (4.2). From this we conclude that indeed the Feynman diagram calculation of the imaginary part does produce the result which is identical to what we obtained from the one-loop effective action of QCD [7, 8].

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7. Schwinger's method

To remove any remaining doubt about the imaginary part (4.2) we now verify the above result by evaluating the functional determinant (3.7) of the effective action perturbatively to the order g^2. With our abelian formalism of QCD, one can do this by following the method Schwinger used to evaluate the effective action of QED perturbatively step by step [13]. This is presented in the appendix. Here we do this in a slightly different way using a modern language, by calculating the functional determinant (3.7) to the order g^2.

To calculate the determinant we consider the determinant of the form

$$\mathrm{Det}\, M = \mathrm{Det}\,(-\hat{D}^2 + gR)\,, \tag{7.1}$$

where $R = \pm 2a/g, \pm 2ib/g$, and 0. We can rewrite

$$\begin{aligned}\mathrm{Det}\, M &= \mathrm{Det}\,(-\partial^2)\cdot \mathrm{Det}\left[1 + \frac{1}{\partial^2} ig(\partial_\mu B_\mu + B_\mu \partial_\mu) - \frac{1}{\partial^2} g^2 B_\mu^2 - \frac{1}{\partial^2} gR\right] \\ &= \mathrm{Det}\,(-\partial^2)\cdot \mathrm{Det}[1 + \Delta + \tilde{\Delta} + \hat{\Delta}]\,,\end{aligned} \tag{7.2}$$

and calculate the determinant by expanding it. At the lowest order in g^2 we have

$$\ln \mathrm{Det} M = \mathrm{Tr} \ln M = \mathrm{Tr}\left(\tilde{\Delta} - \frac{1}{2}\Delta^2 - \frac{1}{2}\hat{\Delta}^2\right). \tag{7.3}$$

Now, using a standard technique [23, 25] (with dimensional regularization) we obtain to the order g^2,

$$\begin{aligned}\mathrm{Tr} \ln (-\hat{D}^2 + gR) &= \frac{ig^2}{16\pi^2} \int d^4p \int_0^1 dv \int_0^\infty \frac{dt}{t} \exp\left[-\frac{p^2}{4\mu^2}(1 - v^2)t\right] \times \\ &\quad \times \left[\frac{v^2}{4} G_{\mu\nu}(p) G_{\mu\nu}(-p) - \frac{1}{2} R(p) R(-p)\right].\end{aligned} \tag{7.4}$$

From this we find that the determinant (3.7) is given, up to the order g^2, by [8]

$$\begin{aligned}\Delta S &= \frac{g^2}{8\pi^2} \int d^4p G_{\mu\nu}(p) G_{\mu\nu}(-p) \Sigma(p)\,, \\ \Sigma(p) &= \int_0^1 dv (1 - \frac{v^2}{4}) \int_0^\infty \frac{dt}{t} \exp[-\frac{p^2}{4\mu^2}(1 - v^2)t] \\ &= -2 \int_0^1 dv \frac{v^2(1 - v^2/12)}{1 - v^2} + C_1\,,\end{aligned} \tag{7.5}$$

where C_1 is a regularization-dependent constant. Now, it is straightforward to evaluate the imaginary part of ΔS, because the imaginary part can only arise from the pole at $v = 1$. So repeating the argument to derive the imaginary part for massless QED in section 5, we again reproduce (4.2), after the proper wave function renormalization [8].

The fact that the Feynman diagram calculation and the Schwinger's method produce an identical result is not accidental, because they are physically equivalent [23]. In fact, one can easily convince oneself that, order by order the perturbative expansion of the determinant (3.7) has one to one correspondence with the Feynman diagram expansion. And up to the order g^2, the Feynman diagrams that contribute to the determinant (3.7) are precisely those given by figure 2. Another way to see this is to notice that, from the definition of the exponential integral function [26]

$$Ei(-z) = -\int_z^\infty \frac{d\tau}{\tau}\exp(-\tau) = \gamma + \ln z + \int_0^z \frac{d\tau}{\tau}\left[\exp(-\tau) - 1\right], \qquad (\mathrm{Re}\, z > 0) \qquad (7.6)$$

we can express $\Sigma(p)$ in (7.5) by

$$\Sigma(p) = -\lim_{\epsilon\to 0}\int_0^1 dv\left(1 - \frac{v^2}{4}\right)\left[\gamma + \ln\left(\frac{p^2}{4\mu^2}(1-v^2)\epsilon\right)\right] = -\frac{11}{12}\left[\ln\left(\frac{p^2}{\mu^2}\right) + C_2\right], \qquad (7.7)$$

where C_2 is another constant which can be removed by the wave function renormalization. But this is exactly what we had in (6.2) in the Feynmam diagram calculation (up to the irrelevant renormalization constant). This confirms that the calculations of the imaginary part by Feynman diagram and by Schwinger's method are indeed physically equivalent.

It must be emphasized again that, just as in the calculation of the non-perturbative effective action, the causality plays the crucial role in these perturbative calculations of the imaginary part of the effective action.

This confirms that the monopole condensation indeed describes a stable vacuum, but the electric background creates the pair-annihilation of the valence gluons in SU(2) *QCD* [7, 8]. As importantly this confirms that the infra-red regularization by causality in the calculation of the effective action of QCD is indeed correct.

8. Discussion

Proof of the monopole condensation in QCD has been extremely difficult to attain. In particular, the instability of the SNO vacuum has created a false impression that one can not prove the monopole condensation by calculating the effective action of QCD. The instability of the SNO vacuum can be traced back to the ζ-function regularization of the infra-red divergence of the gluon loop integral (3.10). But we have recently pointed out that the ζ-function regularization fails to ensure the causality, and proposed the infra-red regularization by causality as the correct regularization in QCD. With the infra-red regularization by causality one can prove that a stable monopole condensation does take place at one loop level which can generate the dual Meissner effect and thus the confinement of color in SU(2) QCD [7, 8].

Given that the ζ-function regularization is a well-established regularization method which has worked so well in other theories, one can not easily dismiss the instability of the SNO vacuum. Under this circumstance one definitely needs to know which regularization in QCD is correct, and why.

In this paper we have checked the correctness of the infra-red regularization by causality with two independent methods based on the perturbative expansion. Both methods produce a result identical to the one obtained with the infra-red regularization by causality, confirming the stability of the monopole condensation. The fact that the perturbative

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method and the infra-red regularization by causality produce an identical result is not surprizing, because the perturbative method is also based on the causality principle. And the causality is what we need to remove the tachyonic modes which causes the instability of the SNO vacuum. This, of course, does not mean that the ζ-function regularization has any intrinsic defect. We emphasize again that the problem was the incorrect inclusion of the tachyonic modes in the calculation of the effective action, not the ζ-function regularization. The ζ-function regularization is simply too honest to correct this mistake.

We emphasize that the above perturbative calculation of the imaginary part of the one-loop effective action was possible because in QCD (and in massless QED) the imaginary part of the one-loop effective action is of the order g^2 [8, 12]. This assures us that one can make a perturbative expansion for the imaginary part of the effective action. For massive QED, for example, this calculation does not make sense because the imaginary (as well as the real) part of the effective action simply does *not* allow a convergent perturbative expansion [10, 11]. The same argument applies to the real part of QCD. Only for the imaginary part of massless gauge theories one can make sense out of the perturbative calculation.

One might worry about the negative signature of the imaginary part in the QCD effective action with the electric background. To understand the origin of this, compare QCD with QED. In QED we integrate the electron loop, and we know that the imaginary part of the effective action is positive [10, 13]. But in QCD we have the valence gluon loop, and obviously the electron and the valence gluon have opposite statistics. Furthermore one can easily convince oneself that the two loop integrals do not change the signature, due to the similar nature of interactions. This means that the imaginary part of the effective action in QED and QCD should have opposite signature. This is the reason for the negative signature in (4.2) [7, 8]. We emphasize that this is what one should have expected from the asymptotic freedom. A positive imaginary part would make pair-creation, not pair-annhilation, of the valence gluons. This would produce screening, not anti-screening, of the color charge. Obviously this is contrary to asymptotic freedom. This confirms that indeed the electric background generates pair-annihilation, not pair-creation, of gluons in QCD.

We have neglected the quarks in this paper. We simply remark that the quarks, just like in asymptotic freedom [24], tend to destabilize the monopole condensation. In fact the stability puts exactly the same constraint on the number of quarks as the asymptotic freedom [27]. Also we have considered only the pure magnetic or pure electric background here. So to be precise, the above result only proves the existence of a stable monopole condensation for a pure magnetic background. To show that this is the true vacuum of QCD, one must calculate the effective action with an arbitrary background in the presence of the quarks and show that the monopole condensation remains a true minimum of the effective potential. Fortunately, one can actually calculate the effective action with an arbitrary constant background, and show that indeed the monopole condensation becomes the true vacuum of SU(2) QCD, at least at one-loop level [27].

The monopole condensation has been established before in a supersymmetric generalization of QCD [28]. But a stable monopole condensation, and by implication the confinement of color, within the framework of QCD has not been demonstrated before. It

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is truly remarkable (and surprising) that the existing principles of quantum field theory allow us to demonstrate the monopole condensation within the framework of QCD.

A. Details of Schwinger's method calculation

With the abelian formalism of QCD, we can follow Schwinger step by step to obtain the imaginary part of the QCD effective action. For the pure magnetic case (i.e., for $b = 0$), (3.7) is written as

$$\Delta S = i\,\mathrm{Tr}[\ln(-\hat{D}^2 + 2a) + \ln(-\hat{D}^2 - 2a)]\,. \tag{A.1}$$

Using the identity

$$\ln M = -\int_0^\infty \frac{ds}{s}\exp(-iMs)\,, \tag{A.2}$$

this can be expressed as

$$\Delta S = -i\,\mathrm{Tr}\int_0^\infty \frac{ds}{s}[\exp(-i(\hat{D}^2 - 2a)s) + \exp(-i(\hat{D}^2 + 2a)s)]\,. \tag{A.3}$$

The perturbative expansion of

$$\mathrm{Tr}\,U(s)_\pm \equiv \mathrm{Tr}\exp(-i(\hat{D}^2 \mp 2a)s)\,, \tag{A.4}$$

in powers of $g\tilde{C}_\mu$ and a (remember that a contains a factor of g) can be read from Schwinger's work [13].

To second order in g we have

$$\mathrm{Tr}\,U(s)_\pm = \mathrm{Tr}\,U_0(s) - is\,\mathrm{Tr}[\mathcal{G}_\pm U_0(s)] - \frac{1}{2}s^2\int_0^1 du\,\mathrm{Tr}[\mathcal{G}_\pm U_0((1-u)s)\mathcal{G}_\pm U_0(us)]\,, \tag{A.5}$$

where

$$U_0(s) = \exp(i\partial^2 s), \mathcal{G}_\pm = -\hat{D}^2 + \partial^2 \pm 2a\,.$$

Substituting these into (A.3) we have

$$\begin{aligned}\Delta S = -2i\int\frac{ds}{s}\Bigg\{&\mathrm{Tr}[\exp(i\partial^2 s)] - is\,\mathrm{Tr}[g^2\tilde{C}_\mu^2\exp(i\partial^2 s)] + \\ &+ g^2s^2\int_{-1}^1 dv\,\mathrm{Tr}\left[\partial_\mu\tilde{C}_\mu\exp\left(\frac{i}{2}\partial^2(1-v)s\right)\partial_\nu\tilde{C}_\nu\exp\left(\frac{i}{2}\partial^2(1+v)s\right)\right] - \\ &- a^2s^2\int_{-1}^1 dv\,\mathrm{Tr}\left[\exp\left(\frac{i}{2}\partial^2(1-v)s\right)\exp\left(\frac{i}{2}\partial^2(1+v)s\right)\right]\Bigg\}\,,\end{aligned} \tag{A.6}$$

where $u = (1+v)/2$. The next step is to take the trace, yielding

$$\begin{aligned}\Delta S = -2i\int\frac{ds}{s}\int d^4p\int\frac{d^4k}{(2\pi)^4}\times \\ \times\Bigg\{\mathrm{Tr}\left[\exp(-ik^2s)\right] - is\,\mathrm{Tr}\left[g^2\tilde{C}_\mu^2\exp(-ik^2s)\right] -\end{aligned}$$

$$- g^2 s^2 \int_{-1}^{1} dv \, \mathrm{Tr} \left[k_\mu \tilde{C}_\mu \exp\left(-\frac{i}{2}\left(k + \frac{p}{2}\right)^2 (1-v)s \right) \times \right.$$
$$\left. \times \, k_\nu \tilde{C}_\nu \exp\left(-\frac{i}{2}\left(k - \frac{p}{2}\right)^2 (1+v)s \right) \right] -$$
$$- a^2 s^2 \int_{-1}^{1} dv \, \mathrm{Tr} \left[\exp\left(-\frac{i}{2}\left(k + \frac{p}{2}\right)^2 (1-v)s \right) \times \right.$$
$$\left. \left. \times \exp\left(-\frac{i}{2}\left(k - \frac{p}{2}\right)^2 (1+v)s \right) \right] \right\} . \tag{A.7}$$

Following Schwinger's technique and keeping only nonvanishing field-dependent terms, we obtain

$$\Delta S = \frac{a^2}{4\pi^2} \int \frac{ds}{s} \int d^4p \int_0^1 dv \left(1 - \frac{v^2}{4}\right) \exp\left(-\frac{i}{4} p^2 (1-v^2)s\right) . \tag{A.8}$$

Now, with the Wick rotation $s \to -is$ and the integration by parts with respect to v, we have

$$\Delta S = \frac{a^2}{8\pi^2} \int d^4p \int \frac{ds}{s} \left\{ \frac{11}{6} + sp^2 \int_0^1 dv v^2 \left(1 - \frac{v^2}{12}\right) \exp\left(-\frac{p^2}{4}(1-v^2)s\right) \right\} . \tag{A.9}$$

The first term in the square bracket has no imaginary part and is removed by renormalization. The second term is integrated over s to yield

$$\Delta S = -\frac{a^2}{2\pi^2} \int d^4p \int_0^1 dv \frac{v^2(1 - v^2/12)}{1 - v^2} , \tag{A.10}$$

which reproduces (7.5) up to the irrelevant renormalization constant.

On the other hand for the pure electric background (i.e., for $a = 0$), we have

$$\Delta S = i \, \mathrm{Tr}[\ln(-\tilde{D}^2 + 2ib) + \ln(-\tilde{D}^2 - 2ib)] . \tag{A.11}$$

Repeating the steps that lead to (A.10) we find

$$\Delta S = \frac{b^2}{2\pi^2} \int d^4p \int_0^1 dv \frac{v^2(1 - v^2/12)}{1 - v^2} . \tag{A.12}$$

So, for an arbitrary background, we have

$$\Delta S = -\frac{a^2 - b^2}{2\pi^2} \int d^4p \int_0^1 dv \frac{v^2(1 - v^2/12)}{1 - v^2} . \tag{A.13}$$

This is identical to (7.5), again up to the irrelevant renormalization constant. This, with the argument in sections 6 and 7, confirms that Schwinger's method reproduces (4.2).

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References

[1] Y. Nambu, *Strings, monopoles and gauge fields*, *Phys. Rev.* **D 10** (1974) 4262;
S. Mandelstam, *Vortices and quark confinement in non-abelian gauge theories*, *Phys. Rept.* **23** (1976) 245;
A.M. Polyakov, *Quark confinement and topology of gauge groups*, *Nucl. Phys.* **B 120** (1977) 429;
G. 't Hooft, *Topology of the gauge condition and new confinement phases in nonabelian gauge theories*, *Nucl. Phys.* **B 190** (1981) 455.

[2] Y.M. Cho, *A restricted gauge theory*, *Phys. Rev.* **D 21** (1980) 1080; *A theory of monopoles*, *J. Korean Phys. Soc.***17** (1984) 266; *Abelian dominance in Wilson loops*, *Phys. Rev.* **D 62** (2000) 074009 [`hep-th/9905127`].

[3] Y.M. Cho, *Glueball spectrum in extended quantum chromodynamics*, *Phys. Rev. Lett.* **46** (1981) 302; *Extended gauge theory and its mass spectrum*, *Phys. Rev.* **D 23** (1981) 2415;
W.S. Bae, Y.M. Cho and S.W. Kimm, *QCD versus Skyrme-Faddeev theory*, *Phys. Rev.* **D 65** (2002) 025005 [`hep-th/0105163`].

[4] G.K. Savvidy, *Infrared instability of the vacuum state of gauge theories and asymptotic freedom*, *Phys. Lett.* **B 71** (1977) 133.

[5] N.K. Nielsen and P. Olesen, *An unstable Yang-Mills field mode*, *Nucl. Phys.* **B 144** (1978) 376;
H.B. Nielsen and P. Olesen, *A quantum liquid model for the QCD vacuum: gauge and rotational invariance of domained and quantized homogeneous color fields*, *Nucl. Phys.* **B 160** (1979) 380;
C. Ragiadakos, *A stable symmetrized Savvidy vacuum*, *Phys. Lett.* **B 100** (1981) 471.

[6] A. Yildiz and P.H. Cox, *Vacuum behavior in quantum chromodynamics*, *Phys. Rev.* **D 21** (1980) 1095;
M. Claudson, A. Yildiz and P.H. Cox, *Vacuum behavior in quantum chromodynamics, II*, *Phys. Rev.* **D 22** (1980) 2022;
S.L. Adler, *Effective action approach to mean field nonabelian statics and a model for bag formation*, *Phys. Rev.* **D 23** (1981) 2905;
W. Dittrich and M. Reuter, *Effective QCD lagrangian with zeta function regularization*, *Phys. Lett.* **B 128** (1983) 321;
M. Reuter and W. Dittrich, *Symmetry restoration by a magnetic field at high temperature*, *Phys. Lett.* **B 144** (1984) 99;
C.A. Flory, *A selfdual gauge field, its quantum fluctuations and interacting fermions*, *Phys. Rev.* **D 28** (1983) 1425;
S.K. Blau, M. Visser and A. Wipf, *Analytical results for the effective action*, *Int. J. Mod. Phys.* **A 6** (1991) 5409;
M. Reuter, M.G. Schmidt and C. Schubert, *Constant external fields in gauge theory and the spin 0, 1/2, 1 path integrals*, *Ann. Phys. (NY)* **259** (1997) 313

[7] Y.M. Cho, H.W. Lee and D.G. Pak, *Faddeev-Niemi conjecture and effective action of QCD*, *Phys. Lett.* **B 525** (2002) 347 [`hep-th/0105198`];
Y.M. Cho and D.G. Pak, *Monopole condensation in* SU(2) *QCD*, *Phys. Rev.* **D 65** (2002) 074027 [`hep-th/0201179`].

[8] Y.M. Cho and M.L. Walker, *Stability of monopole condensation in* SU(2) *QCD*, `hep-th/0206127`.

[9] S.R. Coleman and E. Weinberg, *Radiative corrections as the origin of spontaneous symmetry breaking*, *Phys. Rev.* **D 7** (1973) 1888.

[10] Y.M. Cho and D.G. Pak, *Effective action — a convergent series — of QED*, *Phys. Rev. Lett.* **86** (2001) 1947 [hep-th/0006057];
U.D. Jentschura and E.J. Weniger, *Comment on 'The effective action — a convergent series — of QED' by Y.M. Cho and D.G. Pak (hep-th/0006057)*, *Phys. Rev. Lett.* **88** (2002) 089101 [hep-th/0007108];
Y.M. Cho, *Reply to comment on 'Effective action: a convergent series of QED'*, *Phys. Rev. Lett.* **91** (2003) 039101 [hep-th/0303040].

[11] W.S. Bae, Y.M. Cho and D.G. Pak, *Electric-magnetic duality in QED effective action*, *Phys. Rev.* **D 64** (2001) 017303 [hep-th/0011196];
Y.M. Cho and D.G. Pak, *Renormalization of QED effective action*, hep-th/0010073.

[12] V. Schanbacher, *Gluon propagator and effective lagrangian in QCD*, *Phys. Rev.* **D 26** (1982) 489;
L. Freyhult, *Field decomposition and the ground state structure of* SU(2) *Yang-Mills theory*, *Int. J. Mod. Phys.* **A 17** (2002) 3681 [hep-th/0106239].

[13] J. Schwinger, *On gauge invariance and vacuum polarization*, *Phys. Rev.* **82** (1951) 664.

[14] L.D. Faddeev and A.J. Niemi, *Partially dual variables in* SU(2) *Yang-Mills theory*, *Phys. Rev. Lett.* **82** (1999) 1624 [hep-th/9807069]; *Partial duality in* SU(N) *Yang-Mills theory*, *Phys. Lett.* **B 449** (1999) 214 [hep-th/9812090].

[15] S.V. Shabanov, *An effective action for monopoles and knot solitons in Yang-Mills theory*, *Phys. Lett.* **B 458** (1999) 322 [hep-th/9903223];
S.V. Shabanov, *Yang-Mills theory as an abelian theory without gauge fixing*, *Phys. Lett.* **B 463** (1999) 263 [hep-th/9907182];
H. Gies, *Wilsonian effective action for* SU(2) *Yang-Mills theory with Cho-Faddeev-Niemi-Shabanov decomposition*, *Phys. Rev.* **D 63** (2001) 125023 [hep-th/0102026].

[16] R. Zucchini, *Global aspects of abelian and center projections in* SU(2) *gauge theory*, hep-th/0306287;
K.-I. Kondo, *Magnetic condensation, abelian dominance and instability of savvidy vacuum*, hep-th/0404252.

[17] T.T. Wu and C.N. Yang, *Concept of nonintegrable phase factors and global formulation of gauge fields*, *Phys. Rev.* **D 12** (1975) 3845.

[18] N.S. Manton, *The force between 't Hooft-Polyakov monopoles*, *Nucl. Phys.* **B 126** (1977) 525.

[19] Y.M. Cho, *Colored monopoles*, *Phys. Rev. Lett.* **44** (1980) 1115; *Internal structure of the monopoles*, *Phys. Lett.* **B 115** (1982) 125.

[20] A.A. Belavin, A.M. Polyakov, A.S. Shvarts and Y.S. Tyupkin, *Pseudoparticle solutions of the Yang-Mills equations*, *Phys. Lett.* **B 59** (1975) 85;
G. 't Hooft, *Symmetry breaking through Bell-Jackiw anomalies*, *Phys. Rev. Lett.* **37** (1976) 8.

[21] Y.M. Cho, *Vacuum tunneling in spontaneously broken gauge theory*, *Phys. Lett.* **B 81** (1979) 25.

[22] B. de Witt, *Quantum theory of gravity: the manifestly covariant theory*, *Phys. Rev.* **162** (1967) 1195; *Quantum theory of gravity: applications of the covariant theory*, *Phys. Rev.* **162** (1967) 1239.

[23] See for example, C. Itzykson and J. Zuber, *Quantum field theory*, McGraw-Hill, New York 1980;
M. Peskin and D. Schröeder, *An introduction to quantum field theory*, Addison-Wesley, Reading 1995;
S. Weinberg, *Quantum Theory of Fields*, Cambridge University Press, Cambridge 1996.

[24] D.J. Gross and F. Wilczek, *Ultraviolet behavior of non-abelian gauge theories*, *Phys. Rev. Lett.* **30** (1973) 1343;
H.D. Politzer, *Reliable perturbative results for strong interactions?*, *Phys. Rev. Lett.* **30** (1973) 1346.

[25] A similar calculation was first carried out by Honerkamp. See J. Honerkamp, *The question of invariant renormalizability of the massless Yang-Mills theory in a manifest covariant approach*, *Nucl. Phys.* **B 48** (1972) 269.

[26] See, for example, I. Gradshteyn and I. Ryzhik, *Table of integrals, series and products*, A. Jeffery ed., Academic Press, New York 1994;
M. Abramowitz and I. Stegun, *Handbook of mathematical functions*, Dover 1970.

[27] Y.M. Cho and D.G. Pak, *Dynamical symmetry breaking and magnetic confinement in QCD*, hep-th/0006051; See also, *Dynamical symmetry breaking and magnetic confinement in QCD* in *Proceedings of TMU-Yale Symposium on Dynamics of Gauge Fields*, T. Appelquist and H. Minakata eds., Universal Academy Press, Tokyo 1999.

[28] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation and confinement in $N = 2$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 426** (1994) 19 [hep-th/9407087]; *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, *Nucl. Phys.* **B 431** (1994) 484 [hep-th/9408099].